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Analysis of Quantum Codes over the ring Z_{11} by $(1-2\varsigma)$ – Constacyclic Codes over $Z_{11}+\varsigma Z_{11}$, where $\varsigma^2=\varsigma$

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Abstract: ***This paper is concerned with structural properties and construction of the quantum codes over Z_{11} which were obtained through $(1-2\varsigma)$ - Constacyclic codes over $Z_{11}+\varsigma Z_{11}$, where $\varsigma^2=\varsigma$. A Gray map is defined between $Z_{11}+\varsigma Z_{11}$ and Z^2_{11}. The parameters of Quantum Codes over Z_{11} are obtained by decomposition of $(1-2\varsigma)$ - Constacyclic codes into negacyclic and cyclic codes over Z_{11}. To illustrate results some examples of Quantum Codes of arbitrary length are also obtained.***

Keywords and phrases: Finite ring, Linear codes, Constacyclic codes, Negacyclic codes, Quantum codes.

I. INTRODUCTION

Codes over finite rings were initiated by Blake in early 1970s [13], [14]. Great progress has been made in the 1990s, as the significant discovery that certain good nonlinear binary codes canbe constructed from cyclic codes over Z_4 via the Gray map [15]. There has been an enormous development in the research on quantum codes. As the disclosure that quantum codes secure quantum information similar to classical codes classic information. Quantum information can propagate faster than light under certain conditions, while classical information cannot. Quan- tum information can't be duplicated but classical information can be. Quantum codes provide the most efficient way to overcome decoherence. The first quantum code was found by Shor [9].

After that the construction of quantum codes through classical cyclic codes and their generalizations has developed rapidly. Quantum codes attracted worldwide attention therefore.Later on, Calderbank et al. [3] gave a technique to build quantum codes through classical codes in 1998. Recently the theory of quantum codes is on the path of everlasting development. Inrecent years, the theory of quantum code has been developed rapidly (see reference [4, 6]). A significant development in the construction of quantum codes through cyclic codes over finitechain ring F_2+uF_2, where $u^2=0$ of odd length was made by Qian [7]. Kai and Zhu [5] also gave a method to construct quantum codes through cyclic codes over finite chain ring F_4+uF_4 where $u^2=0$ of odd length. Qian [8] studied quantum codes of arbitrary length through cyclic codesover finite non-chain ring F_2+vF_2, where $v^2=v$. Recently, Ashraf and Mohammad [1] definedthe construction of quantum codes through cyclic codes over finite non-chain ring F_3+vF_3,where $v^2=1$. Then in [2] Ashraf and Mohammad studied this topic over the different finite non-chain ring F_q+vF_q, where $v^2=v$. This motivates us to consider $(1-2\varsigma)$ - Constacyclic codes over the finite non-chain ring $Z_{11}+\varsigma Z_{11}$, where $\varsigma^2=1$ to obtain good quantum codes. The restof the paper is arranged in the following way, Section 2 is Preliminaries in which some funda-mental properties and some essential definitions have been given. In Section 3 Gray Map over $Z_{11}+\varsigma Z_{11}$ and its properties defined. Section 4 presented the development of quantum codesthrough $(1-2\varsigma)$ – constacyclic codes over the considered ring which are exemplified in section4.Finally the paper is concluded in last section.

II. Preliminaries

Let Z_{11} is a finite filed having 11 elements that are {0, 1, 2, 3, ..., 10}. firstly we start with a general overview of the ring

$$\beta = Z_{11}+\varsigma Z_{11} = \{0, 1, 2, ..., 10, \varsigma, 2\varsigma, ..., 10\varsigma, 1+\varsigma, 1+2\varsigma, ..., 1+10\varsigma, 2+\varsigma, 2+2\varsigma, ..., 2+10\varsigma, 3+\varsigma, 3+2\varsigma, ..., 3+10\varsigma, ..., 10+\varsigma, 10+2\varsigma, ..., 10+10\varsigma\}$$

where $\varsigma^2 = \varsigma$, β is a finite, commutative, non-chain, semi-local ring with $11^2 = 121$ elements having characteristic 11.

β has total 100 units which are

$\{1, 2, 3, .., 10, (1+\varsigma), (1+2\varsigma), ..., (1+9\varsigma), (2+\varsigma), (2+2\varsigma), ..., (2+8\varsigma), (2+10\varsigma), (3+\varsigma), (3+2\varsigma), ..., (3+7\varsigma), (3+9\varsigma), (3+10\varsigma)\ (4+\varsigma), (4+2\varsigma), ..., (4+6\varsigma), (4+8\varsigma), ..., (4+10\varsigma), (5+\varsigma), (5+2\varsigma), ..., (5+5\varsigma), (5+7\varsigma), ..., (5+10\varsigma), (6+\varsigma), ..., (6+4\varsigma), (6+6\varsigma), ..., (6+10\varsigma), (7+\varsigma), ...(7+3\varsigma), (7+5\varsigma), ..., (7+10\varsigma), (8+\varsigma), (8+2\varsigma), (8+4\varsigma), ..., (8+10\varsigma), (9+\varsigma), (9+3\varsigma), ..., (9+10\varsigma), (10+2\varsigma), ..., (10+10\varsigma)\}$.

The considered ring β has two maximal ideals which are

$$<\varsigma>$$

and

$$<1-\varsigma>$$

Since, it is clear that $\beta/<\varsigma>$, $\beta/<1-\varsigma>$ both are isomorphic to Z_{11}, that is, $\beta/<\varsigma> \cong Z_{11}$, $\beta/<1-\varsigma> \cong Z_{11}$.

By Chinese remainder theorem, $\beta \cong <\varsigma> \oplus /<1-\varsigma> \cong Z_{11} \oplus Z_{11}$. Therefore, an arbitrary element $\flat_1 + \varsigma\flat_2$ of the considered ring can be written as

$$\flat_1 + \varsigma\flat_2 = (\flat_1 + \flat_2)(\varsigma) + (\flat_1)(1-\varsigma)$$

for all $\flat_1, \flat_2 \in Z_{11}$.

Throughout the paper, we denote units of the ring β as ϑ for sake of simplicity.

A nonempty subset $\mathfrak{J}$ of β^n is called a linear code over β with length n if $\mathfrak{J}$ is an $\beta-$ submodule of β^n and the elements of $\mathfrak{J}$ are called codewords.Let $\mathfrak{J}$ be a code over β with length n and its polynomial representation T ($\mathfrak{J}$) be

$$T(\mathfrak{J}) = \{ \sum_{i=0}^{n-1} \chi_i(\dagger) \mid (\chi_0, \chi_1, ..., \chi_{n-1}) \in \mathfrak{J} \}$$

Let Υ, Λ and $\mho$ are the maps from β^n to β^n defined as

$$\Upsilon(\chi_0,\chi_1,\dots,\chi_{n-1}) = (\chi_{n-1},\chi_0,\dots,\chi_{n-2}),$$

$$\Lambda(\chi_0,\chi_1,\dots,\chi_{n-1}) = (-\chi_{n-1},\chi_0,\dots,\chi_{n-2}),$$

$$\mho(\chi_0,\chi_1,\dots,\chi_{n-1}) = (\varsigma\chi_{n-1},\chi_0,\dots,\chi_{n-2}),$$

respectively. Then $\mathfrak{Z}$ is a cyclic, negacyclic and ϑ-constacyclic if $\Upsilon(\mathfrak{Z}) = \mathfrak{Z}$, $\Lambda(\mathfrak{Z}) = \mathfrak{Z}$ and $\mho(\mathfrak{Z}) = \mathfrak{Z}$ respectively. A code $\mathfrak{Z}$ over β of length n is cyclic, negacyclic and ϑ-constacyclic if and only if $T(\mathfrak{Z})$ is an ideal of $\beta[t]/<(\dagger)^n - 1>, \beta[t]/<(\dagger)^n + 1>$ and $\beta[t]/<(\dagger)^n - \vartheta>$ respectively.

For the arbitrary elements $\chi = (\chi_0,\chi_1,\dots,\chi_{n-1})$ and $\psi = (\psi_0,\psi_1,\dots,\psi_{n-1})$ of β, the inner product is defined as $\chi.\psi = (\chi_0\psi_0 + \chi_1\psi_1 + \dots + \chi_{n-1}\psi_{n-1})$. If $\chi.\psi = 0$, then χ and ψ are orthogonal. If $\mathfrak{Z}$ is a linear code over β of length n, then the dual code of $\mathfrak{Z}$ is defined as $\mathfrak{Z}^\perp = \{\chi \in \beta^n : \chi.\psi = 0 \text{ for all } \psi \in \mathfrak{Z}\}$. which is also a linear code over β of length n. A code $\mathfrak{Z}$ is said to be self orthogonal if $\mathfrak{Z} \subseteq \mathfrak{Z}^\perp$ and said to be self dual if $\mathfrak{Z} = \mathfrak{Z}^\perp$.

III. GRAY MAP OVER β

The map $\varphi: \beta \to Z^2{}_{11}$ described as

$$\varphi(\flat = \wp + \varsigma\wp') = (\wp, \wp + \wp')$$

is the Gray map. This map can be extended to β^n, that is $\varphi: \beta^n \to Z^{2n}{}_{11}$ as
$\varphi(\flat_0,\flat_1,\flat_2,\dots,\flat_{n-1}) = (\wp_0,\wp_0 + \wp'_0,\wp_1,\wp_1 + \wp'_1,\dots,\wp_{n-1},\wp_{n-1} + \wp'_{n-1})$

where $\flat_i = \wp_i + \varsigma\wp'_i$ for all $0 \le i \le n-1$.

Throughout this paper, the code $\mathfrak{Z}$ over β is considered to be with length n.

Proposition 3.1. *The map φ is linear and distance preserving isometry from (β^n, d_L) to $(Z_{11}{}^{2n}, d_H)$.*

Proposition 3.2. *If $\mathfrak{Z}$ is linear self orthogonal code, then so is $\varphi(\mathfrak{Z})$.*

IV. QUANTUM CODES OBTAINED THROUGH $(1-2\Sigma)$ – CONSTACYCLIC CODES OVER $Z_{11} + \Sigma Z_{11}$

Let S, D be two linear codes over β. The operations $\otimes$, $\oplus$ are defined as

$$S \otimes D = \{(s,d) \mid s \in S,\ d \in D\} \quad \text{and} \quad S \oplus D = \{(s+d) \mid s \in S,\ d \in D\}$$

by using properties of Chinese Remainder theorem, any code $\mathfrak{Z}$ over β is permutation equivalent to a code span by the below given matrix:

$$\begin{bmatrix} I_{k_1} & (1-\varsigma)D_1 & \varsigma S_1 & \varsigma S_2 + (1-\varsigma)D_2 & \varsigma S_3 + (1-\varsigma)D_3 \\ 0 & \varsigma I_{k_2} & 0 & \varsigma S_4 & 0 \\ 0 & 0 & (1-\varsigma)I_{k_3} & 0 & (1-\varsigma)D_4 \end{bmatrix}$$

where S_i, D_i are 11-ary matrices for all $1 \le i, j \le 4$.

For a linear code $\mathfrak{Z}$ of length n over β we characterize Let S, D be two linear codes over Z_{11} with length n then, for a linear code $\mathfrak{Z}$, define

$$\mathfrak{Z}_1 = \{a + b \in Z^n_p \mid \text{such that } (a + \varsigma b) \in \mathfrak{Z}\}$$

$$\mathfrak{Z}_2 = \{a \in Z^n_p \mid \text{for some } b \in Z^n_p \text{ such that } (a + \varsigma b) \in \mathfrak{Z}\}$$

are 2, 11-ary codes such that $(1-\varsigma)\,\mathfrak{J}_1 = \mathfrak{J} \bmod (\varsigma)$ and $\varsigma\,\mathfrak{J}_2 = \mathfrak{J} \bmod (1-\varsigma)$. Therefore,

$\mathfrak{J}_1$ and $\mathfrak{J}_2$ are linear codes over Z_{11} having parameters $[n, k_1, d_1]$ and $[n, k_2, d_2]$ respectively. Moreover, the linear code $\mathfrak{J}$ can be uniquely expressed as

$$\mathfrak{J} = \varsigma\,\mathfrak{J}_1 \oplus (1-\varsigma)\,\mathfrak{J}_2$$

and also $|\mathfrak{J}| = |\mathfrak{J}_1||\mathfrak{J}_2|$. Further, $\varphi(\mathfrak{J})$ is a 11-ary linear $[2n, k_1 + k_2, \min(d_1, d_2)]$ code.

Proposition 4.1. *Let* $\mathfrak{J} = \varsigma\mathfrak{J}_1 \oplus (1-\varsigma)\mathfrak{J}_2$ *be a linear code over* β *of length* n *such that* $\mathfrak{J}_1$ *be a linear code having parameters* $[n, k_1, d_1]$ *and* $\mathfrak{J}_2$ *be a linear code having parameters* $[n, k_2, d_2]$. *Then* $\varphi(\mathfrak{J})$ *is a q-ary linear code having parameters* $[2n, k_1 + k_2, \min(d_1, d_2)]$.

Theorem 4.2. *For* $\vartheta = 1 - 2\varsigma$, *the code* $\mathfrak{J}$ *is a* ϑ*-constacyclic code over* β *if and only if* $\mathfrak{J}_1$ *is negacyclic code and* $\mathfrak{J}_2$ *is cyclic code over* Z_{11}.

Proof. For any

$$\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{n-1}) \in \beta^n,$$

where $\zeta_i = \varsigma\breve{f}_i + (1-\varsigma)\breve{l}_i$, with $\breve{f} = (\breve{f}_0, \breve{f}_1, \ldots, \breve{f}_{n-1}) \in \mathfrak{J}_1$ and $\breve{l} = (\breve{l}_0, \breve{l}_1, \ldots, \breve{l}_{n-1}) \in \mathfrak{J}_2$ for all

$\breve{f}_i, \breve{l}_i \in Z_{11}$, $0 \le i \le n-1$.

For $(1-2\varsigma)$ - constacyclic code $\mathfrak{J}$

$$\begin{aligned}
\mho(\chi) &= ((1-2\varsigma)\zeta_{n-1}, \zeta_0, \ldots, \zeta_{n-2}) \\
&= ((\varsigma - 2\varsigma^2)\breve{f}_{n-1} + (1-\varsigma)\breve{l}_{n-1}, \varsigma\breve{f}_0 + (1-\varsigma)\breve{l}_0, \ldots, \varsigma\breve{f}_{n-2} + (1-\varsigma)\breve{l}_{n-2}) \\
&= (-\varsigma\breve{f}_{n-1} + (1-\varsigma)\breve{l}_{n-1}, \varsigma\breve{f}_0 + (1-\varsigma)\breve{l}_0, \ldots, \varsigma\breve{f}_{n-2} + (1-\varsigma)\breve{l}_{n-2}) \\
&= \varsigma\Lambda(\breve{f}) + (1-\varsigma)\Upsilon(\breve{l})
\end{aligned}$$

which in $\mathfrak{J}$. Therefore, $\mathfrak{J}_1$ is a negacyclic and $\mathfrak{J}_2$ is a cyclic codes over the Z_{11} with length n.

Again, $\mathfrak{J}_1$ is a negacyclic codes and $\mathfrak{J}_2$ is a cyclic codes over the Z_{11} with length n, then

$$\Lambda(\breve{f}) = \Lambda(\breve{f}_0, \breve{f}_1, \ldots, \breve{f}_{n-1}) = (-\breve{f}_{n-1}, \breve{f}_0, \ldots, \breve{f}_{n-2}) \in \mathfrak{J}_1$$

and $\Upsilon(\breve{l}) = (\breve{l}_0, \breve{l}_1, \ldots, \breve{l}_{n-1}) = (\breve{l}_{n-1}, \breve{l}_0, \ldots, \breve{l}_{n-2}) \in \mathfrak{J}_2$.

Hence, we have $\varsigma\Lambda(\breve{f}) + (1-\varsigma)\Upsilon(\breve{l}) \in \varsigma\,\mathfrak{J}_1 \oplus (1-\varsigma)\,\mathfrak{J}_2 = \mathfrak{J}$, implies

$$\begin{aligned}
&\varsigma(-\breve{f}_{n-1}, \breve{f}_0, \ldots, \breve{f}_{n-2}) + (1-\varsigma)\,(\breve{l}_{n-1}, \breve{l}_0, \ldots, \breve{l}_{n-2}) \\
&= ((1-2\varsigma)\,(\varsigma\breve{f}_{n-1} + (1-\varsigma)\breve{l}_{n-1}), \varsigma\breve{f}_0 + (1-\varsigma)\breve{l}_0, \ldots, \varsigma\breve{f}_{n-2} + (1-\varsigma)\breve{l}_{n-2}) \\
&= ((1-2\varsigma)\,\zeta_{n-1}, \zeta_0, \ldots, \zeta_{n-2}) \\
&= \mho(\zeta_0, \zeta_1, \ldots, \zeta_{n-1}) = \mho(\chi) \in \mathfrak{J}.
\end{aligned}$$

Hence, $\mathfrak{J}$ is a $(1-2\varsigma)$-constacyclic codes over β.

Theorem 4.3. *For a* $1-2\varsigma$*-constacyclic code,*

$$\mathfrak{J} = <\varsigma\hbar_1(\dagger),(1-\varsigma)\hbar_2(\dagger)> = <\varsigma\hbar_1(\dagger)+(1-\varsigma)\hbar_2(\dagger)>,$$

with $|\mathfrak{J}| = 11^{2n-\deg(\hbar_1(\dagger))-\deg(\hbar_2(\dagger))}$, where $\hbar_i(\dagger)$ *for* $i = 1,2$ *denotes the generator polynomial of* $\mathfrak{J}_i$ *for* i *=1,2.*

Lemma 4.4. *Let* $\mathfrak{J}$ *be a* $(1-2\varsigma)$*-constacyclic codes over* β*. Then*

$$\mathfrak{J} = <\varsigma\hbar_1(\dagger),(1-\varsigma)\hbar_2(\dagger)> = <\varsigma\hbar_1(\dagger)+(1-\varsigma)\hbar_2(\dagger)>$$

with $|\mathfrak{J}| = 11^{2n-\deg(\hbar_1(\dagger))-\deg(\hbar_2(\dagger))}$

where $g_i(t)$ *for* $i = 1,2$ *are the generator polynomials of* $\mathfrak{J}_1$ *and* $\mathfrak{J}_2$ *respectively.*

Theorem 4.5. *Dual of a* $(1-2\varsigma)$ *- constacyclic code is of similar length* $(1-2\varsigma)$ *- constacyclic code.*

Proof. The proof holds trivially as,

$$(1-2\varsigma)^{-1} = (1-2\varsigma).$$

Lemma 4.6. *For a* $(1-2\varsigma)$ *- constacyclic code, the dual code*

1. $\mathfrak{J}^{\perp} = \varsigma\mathfrak{J}_1^{\perp} \oplus (1-\varsigma)\mathfrak{J}_2^{\perp}$
2. $\mathfrak{J}^{\perp} = <\varsigma\hbar_1^{*}(\dagger),(1-\varsigma)\hbar_2^{*}(\dagger)>$ 1
3. $|\mathfrak{J}^{\perp}| = 11^{\deg(\hbar_1(\dagger))+\deg(\hbar_2(\dagger))}$

where, $\hbar^{\star}(\dagger)$ *are reciprocal polynomials.*

Lemma 4.7. *[3] If* $\mathfrak{J}$ *is a cyclic or negacyclic code over* Z_{11} *with generator polynomial g(†), then* $\mathfrak{J}^{\perp} \in \mathfrak{J}$ *if and only if* $\dagger^n - \iota \equiv 0 \mod(g(\dagger)g^{\star}(\dagger))$*,where* $\iota = \pm 1$.

Theorem 4.8. *If* $\mathfrak{J} = <\varsigma\hbar_1(\dagger),(1-\varsigma)\hbar_2(\dagger)>$ *is a* $(1-2\varsigma)$ *- constacyclic codes over* β *, then*$_{\text{¸}}{}^{\perp} \subseteq \mathfrak{J}$ *if and only if*

$$\dagger^n + 1 \equiv 0 \mod(\hbar_1(\dagger)\hbar_1^{\star}(\dagger)) \text{ for } \mathfrak{J}_1$$

and

$$\dagger^n - 1 \equiv 0 \mod(\hbar_2(\dagger)\hbar_2(\dagger)) \text{ for } \mathfrak{J}_2.$$

Proof. First consider

$$\dagger \;\; + 1 \equiv 0 \mod(\hbar_1(\dagger)\hbar^{\star}(\dagger)) \text{ for } \mathfrak{J}_1,$$

$$\dagger^n - 1 \equiv 0 \mod(\hbar_2(\dagger)\hbar^{*}{}_2(\dagger)) \text{ for } \mathfrak{J}_2.$$

Then, due to Lemma 4.7,and, therefore,

$$\mathfrak{J}_1^{\perp} \subseteq \mathfrak{J}_1 \text{ and } \mathfrak{J}_2^{\perp} \subseteq \mathfrak{J}_2$$

$$\varsigma\mathfrak{J}_1^{\perp} \subseteq \varsigma\mathfrak{J}_1, (1-\varsigma)\mathfrak{J}_2^{\perp} \subseteq (1-\varsigma)\mathfrak{J}_2,$$

which in turn implies

$$\varsigma\mathfrak{J}_1^{\perp} \oplus (1-\varsigma)\mathfrak{J}_2^{\perp} \subseteq \varsigma\mathfrak{J}_1 \oplus (1-\varsigma)\mathfrak{J}_2.$$

Thus,

$$\varsigma\mathfrak{J}_1^{\perp} \oplus (1-\varsigma)\mathfrak{J}_2^{\perp} \subseteq \varsigma\mathfrak{J}_1 \oplus (1-\varsigma)\mathfrak{J}_2.$$

$$< \varsigma\hbar_1^{\star}(\dagger) + (1-\varsigma)\hbar_2^{\star}(\dagger) > \subseteq < \varsigma\hbar_1(\dagger) + (1-\varsigma)\hbar_2(\dagger) >$$

and, hence

$$\mathfrak{J}^{\perp} \subseteq \mathfrak{J}.$$

Conversely, consider

$$\mathfrak{J}^{\perp} \subseteq \mathfrak{J},$$

then

$$\varsigma\mathfrak{J}_1^{\perp} \oplus (1-\varsigma)\mathfrak{J}_2^{\perp} \subseteq \varsigma\mathfrak{J}_1 \oplus (1-\varsigma)\mathfrak{J}_2,$$

which implies

$$\varsigma\mathfrak{J}_1^{\perp} \subseteq \varsigma\mathfrak{J}_1 \text{ and } (1-\varsigma)\mathfrak{J}_2^{\perp} \subseteq (1-\varsigma)\mathfrak{J}_2,$$

Hence

$$\mathfrak{J}_1^{\perp} \subseteq \mathfrak{J}_1 \text{ and } \mathfrak{J}_2^{\perp} \subseteq \mathfrak{J}_2,$$

due to Lemma 4.7

$$\dagger^{n} + 1 \equiv 0 \bmod(\hbar_1(\dagger)\hbar_1^{\star}(\dagger)) \text{ for } \mathfrak{J}_1$$

and

$$\dagger^{n} - 1 \equiv 0 \bmod(\hbar_2(\dagger)\hbar_2^{\star}(\dagger)) \text{ for } \mathfrak{J}_2.$$

Theorem 4.9. *Let $\mathfrak{J}$ be a ζ-constacyclic codes over β. Then it's dual code $\mathfrak{J}^{\perp}$ is also a ζ^{-1}-constacyclic codes over β of length* n.

Corollary 4.10. *If code $\mathfrak{J} = \varsigma\mathfrak{J}_1 \oplus (1-\varsigma)\mathfrak{J}_2$ is a $(1-2\varsigma)$ - constacyclic codes over β. Then $\mathfrak{J}^{\perp} \subseteq \mathfrak{J}$ if and only if $\mathfrak{J}_1^{\perp} \subseteq \mathfrak{J}_1$ and $\mathfrak{J}_2^{\perp} \subseteq \mathfrak{J}_2$.*

Lemma 4.11. *[11] (CSS Construction). Let $\mathfrak{J}$ be a linear code over the ring Z_{11} having parameters* [n, k, d]. *Then a quantum code having parameter $[n, 2k-n, \geq d]_{11}$ can be obtainedif*

$\mathfrak{J}^{\perp} \subseteq \mathfrak{J}$.

Quantum codes can be constructed by using above Corollary 4.10 and Lemma 4.11 as follow

Theorem 4.12. *If $\mathfrak{J} = \varsigma\mathfrak{J}_1 \oplus (1-\varsigma)\mathfrak{J}_2 = < \varsigma\hbar_1(\dagger) + (1-\varsigma)\hbar_2(\dagger) >$ is a $(1-2\varsigma)$ -constacyclic codes over β of length n where $g_i(\dagger)$ are generator polynomials of $\mathfrak{J}_1$ and $\mathfrak{J}_2$ for $i = 1,2$ respectively. If $\mathfrak{J}_1^{\perp} \subseteq \mathfrak{J}_1$ and $\mathfrak{J}_2^{\perp} \subseteq \mathfrak{J}_2$.Then there exists a quantum code having parameters*

$[2n, 2k-2n, \geq d_L]_{11}$ *where k is the dimension of linear code $\varphi(\mathfrak{J})$ and d_L is minimum Lee distance of $\mathfrak{J}$.*

V. EXAMPLE

To illustrate the results and existence of quantum codes through $(1-2\varsigma)$ - constacyclic codes over β examples are discussed in this section.

Example 5.1. *In* $Z_{11}(\dagger), \dagger^5 + 1 = (\dagger + 1)(\dagger + 3)(\dagger + 4)(\dagger + 5)(\dagger + 9)$ *and* $\dagger^5 - 1 = (\dagger + 2)(\dagger + 6)(\dagger + 7)(\dagger + 8)(\dagger + 10)$ *Let* $\mathfrak{Z}$ *be a* $(1 - 2\varsigma)$ *- constacyclic codes over* $\beta = Z_{11} + \varsigma Z_{11}$ *where* $\varsigma^2 = \varsigma$ *with length 5.*

Let $\hbar_1(\dagger) = (\dagger + 3)$ *and* $\hbar_2(\dagger) = (\dagger + 7)$ *then* $g(\dagger) = \varsigma(\dagger + 3) + (1 - \varsigma)(\dagger + 7)$ *be the generator polynomial of* $\mathfrak{Z}$.

Since $\hbar_1(\dagger)\hbar_1^*(\dagger)/(\dagger^5 + 1)$, $\hbar_2(\dagger)\hbar_2^*(\dagger)/(\dagger^5 - 1)$ *then by using Theorem 4.8, we get* $\mathfrak{Z}^\perp \subseteq \mathfrak{Z}$.

Further $\varphi(\mathfrak{Z})$ *is a* $[10, 8, 2]$ *linear code and by using Theorem 4.12, the quantum codes having parameters* $[10, 6, \geq 2]_{11}$ *are obtained.*

Example 5.2. *In* $Z_{11}(\dagger), \dagger^{10} + 1 = (\dagger^2 + 1)(\dagger^2 + 3)(\dagger^2 + 4)(\dagger^2 + 5)(\dagger^2 + 9)$ *and* $\dagger^{10} - 1 = (\dagger + 1)(\dagger + 2)(\dagger + 3)(\dagger + 4)(\dagger + 5)(\dagger + 6)(\dagger + 7)(\dagger + 8)(\dagger + 9)(\dagger + 10)$ *Let* $\mathfrak{Z}$ *be a* $(1 - 2\varsigma)$ *-*

constacyclic codes over $\beta = Z_{11} + \varsigma Z_{11}$ *where* $\varsigma^2 = \varsigma$ *of length 10.*

Let $\hbar_1(\dagger) = (\dagger^2 + 3)$ *and* $\hbar_2(\dagger) = (\dagger + 4)$ *then* $g(\dagger) = \varsigma(\dagger^2 + 3) + (1 - \varsigma)(\dagger + 4)$ *be the generator polynomial of* $\mathfrak{Z}$.

Since $\hbar_1(\dagger)\hbar_1^*(\dagger)/(\dagger^{10} + 1)$, $\hbar_2(\dagger)\hbar_2^*(\dagger)/(\dagger^{10} - 1)$ *then by using Theorem 4.8, we get* $\mathfrak{Z}^\perp \subseteq \mathfrak{Z}$.

Further $\varphi(\mathfrak{Z})$ *is a* $[20,17,3]$ *linear code and by using Theorem 4.12, quantum codes having parameters* $[20, 14, \geq 3]_{11}$ *are obtained.*

Example 5.3. *In* $Z_{11}(\dagger), \dagger^{11} + 1 = (1 + \dagger)^{11}$ *and* $\dagger^{11} - 1 = (10 + \dagger)^{11}$ *Let* $\mathfrak{Z}$ *be a* $(1 - 2\varsigma)$ *- constacyclic codes over* $\beta = Z_{11} + \varsigma Z_{11}$ *where* $\varsigma^2 = \varsigma$ *of length 11.*

Let $\hbar_1(\dagger) = (1 + \dagger)^2$ and $\hbar_2(\dagger) = (10 + \dagger)^2$ *then* $g(\dagger) = \varsigma(\dagger + 1)^2 + (1 - \varsigma)(\dagger + 10)^2$ *be the generator polynomial of* $\mathfrak{Z}$.

Since $\hbar_1(\dagger)\hbar_1^*(\dagger)/(\dagger^{11} + 1)$, $\hbar_2(\dagger)\hbar_2^*(\dagger)/(\dagger^{11} - 1)$ *then by using Theorem 4.8, we get* $\mathfrak{Z}^\perp \subseteq \mathfrak{Z}$.

Further $\varphi(\mathfrak{Z})$ *is a* $[22,18,3]$ *linear code by using Theorem 4.12, quantum codes having parameters* $[22, 14, \geq 3]_{11}$ *are obtained.*

Example 5.4. *In* $Z_{11}(\dagger), \dagger^{15} + 1 = (1 + \dagger)(3 + \dagger)(4 + \dagger)(5 + \dagger)(9 + \dagger)(4 + 2\dagger + \dagger^2)(3 + 6\dagger + \dagger^2)(5 + 7\dagger + \dagger^2)(9 + 8\dagger + \dagger^2)(1 + 10\dagger + \dagger^2)$ *and* $\dagger^{15} - 1 = (2 + \dagger)(6 + \dagger)(7 + \dagger)(8 + \dagger)(10 + \dagger)(1 + \dagger + \dagger^2)(9 + 3\dagger + \dagger^2)(5 + 4\dagger + \dagger^2)(3 + 5\dagger + \dagger^2)(4 + 9\dagger + \dagger^2)$. *Let* $\mathfrak{Z}$ *be a* $(1 - 2\varsigma)$ *- constacyclic codes over* $\beta = Z_{11} + \varsigma Z_{11}$ *where* $\varsigma^2 = \varsigma$ *of length 15.*

Let $\hbar_1(\dagger) = (\dagger^2 + 2\dagger + 4)$ *and* $\hbar_2(\dagger) = (\dagger^2 + 4\dagger + 5)$ *then* $g(\dagger) = \varsigma(\dagger + 3) + (1 - \varsigma)(\dagger + 7)$ *be the generator polynomial of* $\mathfrak{Z}$.

Since $\hbar_1(\dagger)\hbar_1^*(\dagger)/(\dagger^{15} + 1)$, $\hbar_2(\dagger)\hbar_2^*(\dagger)/(\dagger^{15} - 1)$ *then by using Theorem 4.8, we get* $\mathfrak{Z}^\perp \subseteq \mathfrak{Z}$. *Further* $\varphi(\mathfrak{Z})$ *is a* $[30,26,3]$ *linear code and by using Theorem 4.12, quantum codes having parameters* $[30, 22, \geq 3]_{11}$ *are obtained.*

Example 5.5. *In* $Z_{11}(\dagger), \dagger^{20} + 1 = (6 + \dagger + \dagger^2)(2 + 2\dagger + \dagger^2)(10 + 3\dagger + \dagger^2)(8 + 4\dagger + \dagger^2)(7 + 5\dagger +$

$\dagger^2)(7+6\dagger+\dagger^2)(8+7\dagger+\dagger^2)(10+8\dagger+\dagger^2)(2+9\dagger+\dagger^2)(6+10\dagger+\dagger^2)$ *and* $\dagger^{20}-1 = (1+\dagger)(2+\dagger)(3+\dagger)(4+\dagger)(5+\dagger)(6+\dagger)(7+\dagger)(8+\dagger)(9+\dagger)(10+\dagger)(1+\dagger^2)(3+\dagger^2)(4+\dagger^2)(5+\dagger^2)(9+\dagger^2)$

Let $\mathfrak{Z}$ *be a* $(1-2\varsigma)$ *- constacyclic codes over* $\beta = Z_{11}+\varsigma Z_{11}$ *where* $\varsigma^2=\varsigma$ *of length 20.*

Let $\hbar_1(\dagger) = (\dagger+3)$ *and* $\hbar_2(\dagger) = (\dagger+7)$ then $g(\dagger) = \varsigma(\dagger+3)+(1-\varsigma)(\dagger+7)$ *be the generator polynomial of* $\mathfrak{Z}$.

Since $\hbar_1(\dagger)\hbar_1^*(\dagger)/(\dagger^5+1), \hbar_2(\dagger)\hbar_2^*(\dagger)/(\dagger^5-1)$ *then by using Theorem 4.8, we get* $\mathfrak{Z}^\perp \subseteq \mathfrak{Z}$.

Further $\varphi(K)$ *is a* $[40,38,2]$ *linear code and by using Theorem 4.12, quantum codes havingparameters* $[40, 36, \geq 2]_{11}$ *are obtained.*

Example 5.6. *In* $Z_{11}(\dagger), \dagger^{22}+1 = (1+\dagger^2)^{11}$ *and* $\dagger^{22}-1 = (1+\dagger)^{11}(10+\dagger)^{11}$ *Let* $\mathfrak{Z}$ *be a* $(1-2\varsigma)$ *- constacyclic codes over* $\beta = Z_{11}+\varsigma Z_{11}$ *where* $\varsigma^2=\varsigma$ *of length 22.*

Let $\hbar_1(\dagger) = (\dagger^2+1)$ *and* $\hbar_2(\dagger) = (\dagger+10)$ *then* $g(\dagger) = \varsigma(\dagger+3)+(1-\varsigma)(\dagger+7)$ *be the generator polynomial of* $\mathfrak{Z}$.

Since $\hbar_1(\dagger)\hbar_1^*(\dagger)/(\dagger^{22}+1), \hbar_2(\dagger)\hbar_2^*(\dagger)/(\dagger^{22}-1)$ *then by using Theorem 4.8, we get* $\mathfrak{Z}^\perp \subseteq \mathfrak{Z}$.

Further $\varphi(K)$ *is a* $[44,41,3]$ *linear code and by using Theorem 4.12, quantum codes havingparameters* $[44,38, \geq 3]_{11}$ *are obtained.*

Example 5.7. *In* $Z_{11}(\dagger), \dagger^{30}+1 = (1+\dagger^2)(3+\dagger^2)(4+\dagger^2)(5+\dagger^2)(9+\dagger^2)(4+\dagger+\dagger^2)(5+2\dagger+\dagger^2)(3+3\dagger+\dagger^2)(9+4\dagger+\dagger^2)(1+5\dagger+\dagger^2)(1+6\dagger+\dagger^2)(9+7\dagger+\dagger^2)(3+8\dagger+\dagger^2)(5+9\dagger+\dagger^2)(4+10\dagger+\dagger^2)$ *and*

$\dagger^{30}-1 = (1+\dagger)(2+\dagger)(3+\dagger)(4+\dagger)(5+\dagger)(6+\dagger)(7+\dagger)(8+\dagger)(9+t)(10+\dagger)(1+\dagger+\dagger^2)(4+2\dagger+\dagger^2)(9+3\dagger+\dagger^2)(5+4\dagger+\dagger^2)(3+5\dagger+\dagger^2)(3+6\dagger+\dagger^2)(5+7\dagger+\dagger^2)(9+8\dagger+\dagger^2)(4+9\dagger+(\dagger)^{[}2])(1+10\dagger+\dagger^2)$

Let $\mathfrak{Z}$ *be a* $(1-2\varsigma)$ *- constacyclic codes over* $\beta = Z_{11}+\varsigma Z_{11}$ *where* $\varsigma^2=\varsigma$ *of length 30.*

Let $\hbar_1(\dagger) = (\dagger^2+9)$ *and* $\hbar_2(\dagger) = (\dagger+7)$ then $g(\dagger) = \varsigma(\dagger+3)+(1-\varsigma)(\dagger+7)$ *be the generator polynomial of* $\mathfrak{Z}$.

Since $\hbar_1(\dagger)\hbar_1^*(\dagger)/(\dagger^{30}+1), \hbar_2(\dagger)\hbar_2(\dagger)/(\dagger^{30}-1)$ *then by using Theorem 4.8, we get* $\mathfrak{Z}^\perp \subseteq \mathfrak{Z}$. *Further* $\varphi(K)$ *is a* $[60,57,3]$ *linear code and by using Theorem 4.12, quantum codes havingparameters* $[60,54, \geq 3]_{11}$ *are obtained*

Table: Some examples of Quantum codes with different parameter.

n	Generator Polynomials	$\varphi(\mathfrak{Z})$	$[\mathfrak{n}\ \mathfrak{k}\ \mathfrak{d}\ ]$
4	$\hbar_1(\dagger) = \dagger^2+3\dagger+10$ and $\hbar_2(\dagger) = \dagger+1$	[8,5,3]	$[8,2,\geq 3]$
7	$\hbar_1(\dagger) = \dagger+1$ and $\hbar_2(\dagger) = \dagger+10$	[14,12,2]	$[14,10,\geq 2]$
12	$\hbar_1(\dagger) = \dagger^2+2\dagger+10$ and $\hbar_2(\dagger) = \dagger+1$	[24,21,3]	$[24,18,\geq 3]$
13	$\hbar_1(\dagger) = \dagger+1$ and $\hbar_2(\dagger) = \dagger+10$	[26,24,2]	$[26,22,\geq 2]$

17	$\hbar_1(\dagger) = \dagger + 1$ and $\hbar_2(\dagger) = \dagger + 10$	[34,32,2]	$[34, 30, \geq 2]$
18	$\hbar_1(\dagger) = \dagger^2 + 1$ and $\hbar_2(\dagger) = \dagger^2 + 10\dagger + 1$	[36,32,3]	$[36, 28, \geq 3]$
21	$\hbar_1(\dagger) = \dagger^2 + 10\dagger + 1$ and $\hbar_2(\dagger) = \dagger + 10$	[42,39,3]	$[42, 36, \geq 3]$
23	$\hbar_1(\dagger) = \dagger + 1$ and $\hbar_2(\dagger) = \dagger + 10$	[46,44,2]	$[46, 42, \geq 2]$

VI. Conclusions

In this paper, we have given a construction for quantum codes through $(1-2\varsigma)$ - constacyclic codes over the finite non-chain ring $\beta = Z_{11} + \varsigma Z_{11}$ where $\varsigma^2 = \varsigma$. We have derived self-orthogonal codes over the ring Z_{11} as Gray images of linear codes over the ring $\beta = Z_{11} + \varsigma Z_{11}$. In particular,the parameters of quantum codes over the ring Z_{11} are obtained by decomposing $(1-2\varsigma)$ - constacyclic codes into cyclic and negacyclic codes over the ring Z_{11}.For the further scope, one can look at other classes of constacyclic codes over β and $\Re = Z_p + \varsigma Z_p$ where $\varsigma^2 = \varsigma$

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